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AN EXACT SOLUTION TO AN EQUATION AND THE FIRST EIGENVALUE OF A COMPACT MANIFOLD

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Abstract. We study an exact solution to a singular ordinary differential equation and use the solution to give a new estimate on the lower bound of the first non-zero eigenvalue of a closed Riemannian manifold with a negative lower bound on the Ricci curvature in terms of the lower bound on the Ricci curvature and the largest interior radius of the nodal domains of the eigenfunction. This provides a new way to estimate eigenvalues.

1. Introduction

Spectral geometry has had an impact on many developments of mathematics. There have been numerous studies on eigenvalues, and especially the first non-zero eigenvalue, of Riemannian manifolds. While there are many results on the first non-zero eigenvalue for closed manifolds with positive Ricci curvature, there are only a few results for manifolds with negative lower bound on the Ricci curvature. In this paper, we study such problems. Let us recall some previous results. Li and Yau [2] proved that for an n-dimensional closed Riemannian manifold with Ricci curvature bounded below by a constant $(n-1)\kappa < 0$ the first non-zero eigenvalue λ of the Laplacian has the lower bound

$$\lambda \geq \frac{1}{2(n-1)d^2 e^{1+\sqrt{1+4(n-1)^2 d^2 |\kappa|}}},$$

where d is the diameter of the manifold. H. C. Yang [4] obtained a better estimate,

$$\lambda \geq \frac{\pi^2}{d^2 e^{\max\{\sqrt{n-1},\sqrt{2}\}\sqrt{(n-1)|\kappa| d^2}}}.$$

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In this paper, we give a new estimate on the lower bound. Note that the interior radius of a domain in a manifold M is the radius of the largest ball contained in the domain. We have the following result.

THEOREM 1.1. *If* (M^n, g) *is an* n*-dimensional closed Riemannian manifold and if the Ricci curvature* $\mathrm{Ric}(M)$ *of* (M^n, g) *has a lower bound* $(n-1)\kappa < 0$*, that is,*

$$R_{ij} \geq \kappa \delta_{ij}, \tag{1}$$

then the first non-zero eigenvalue λ *of the Laplacian* Δ *of* (M^n, g) *satisfies the inequality*

$$\lambda \geq \frac{\pi^2}{d^2[1-(n-1)\kappa/(2\lambda)]} > 0$$

and λ *has the lower bound*

$$\lambda \geq \frac{\pi^2}{d^2} + \frac{1}{2}(n-1)\kappa, \tag{2}$$

where $d = 2r$*, and* r *is the largest interior radius of the nodal domains of the first eigenfunction.*

In the next section, we study the properties of an exact solution ξ, which the author constructed in [3], to a singular ordinary differential equation. In the final section we use this solution and its properties and the structure of the nodal domains of the eigenfunction to derive our estimate for the first non-zero eigenvalue. This provides a new way to estimate eigenvalues.

2. An exact solution to a differential equation

THEOREM 2.1. *The function*

$$\xi(t) = \frac{\cos^2 t + 2t \sin t \cos t + t^2 - \frac{\pi^2}{4}}{\cos^2 t} \quad on \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \tag{3}$$

is an exact solution of the equation

$$\frac{1}{2}\xi'' \cos^2 t - \xi' \cos t \sin t - \xi = 2\cos^2 t \quad in \ \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{4}$$

Moreover, the function ξ *has the following properties:*

$$\xi' \cos t - 2\xi \sin t = 4t \cos t, \tag{5}$$

$$\int_0^{\frac{\pi}{2}} \xi(t)\, dt = -\frac{\pi}{2}, \tag{6}$$

$$
\begin{aligned}
&1-\frac{\pi^2}{4}=\xi(0)\le\xi(t)\le\xi(\pm\frac{\pi}{2})=0 \quad \mathit{on}\ \left[-\frac{\pi}{2},\frac{\pi}{2}\right],\\
&\xi' \ \mathit{is\ increasing\ on}\ \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \ \mathit{and}\ \xi'(\pm\frac{\pi}{2})=\pm\frac{2\pi}{3},\\
&\xi'(t)<0\ \mathit{on}\ \left(-\frac{\pi}{2},0\right)\ \mathit{and}\ \ \xi'(t)>0\ \mathit{on}\ \left(0,\frac{\pi}{2}\right),\\
&\xi''(\pm\frac{\pi}{2})=2,\ \xi''(0)=2\left(3-\frac{\pi^2}{4}\right)\ \mathit{and}\ \ \xi''(t)>0\ \mathit{on}\ \left[-\frac{\pi}{2},\frac{\pi}{2}\right],\\
&\left(\frac{\xi'(t)}{t}\right)'>0\ \mathit{on}\ \left(0,\frac{\pi}{2}\right)\ \mathit{and}\ 2\left(3-\frac{\pi^2}{4}\right)\le\frac{\xi'(t)}{t}\le\frac{4}{3}\ \mathit{on}\ \left[-\frac{\pi}{2},\frac{\pi}{2}\right],\\
&\xi'''\left(\frac{\pi}{2}\right)=\frac{8\pi}{15},\xi'''(t)<0\ \mathit{on}\ \left(-\frac{\pi}{2},0\right)\ \mathit{and}\ \ \xi'''(t)>0\ \mathit{on}\ \left(0,\frac{\pi}{2}\right).
\end{aligned}
$$

Proof of Theorem 2.1. For convenience, let $q(t)=\xi'(t)$, that is,

$$
q(t)=\xi'(t)=\frac{2\left(2t\cos t+t^2\sin t+\cos^2 t\sin t-\frac{\pi^2}{4}\sin t\right)}{\cos^3 t}. \tag{7}
$$

Equation (4), the values $\xi(\pm\pi/2)=0$, $\xi(0)=1-\pi^2/4$ and $\xi'(\pm\pi/2)=\pm 2\pi/3$ can be verified directly from (3) and (7) . The values of ξ'' at 0 and $\pm\pi/2$ can be computed via (4).

(5) is equivalent to $(\xi(t)\cos^2 t)'=4t\cos^2 t$. Therefore

$$
\xi(t)\cos^2 t=\int_{\frac{\pi}{2}}^{t} 4s\cos^2 s\,ds,
$$

and

$$
\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\xi(t)\,dt&=2\int_0^{\frac{\pi}{2}}\xi(t)\,dt\\
&=-8\int_0^{\frac{\pi}{2}}\left(\frac{1}{\cos^2(t)}\int_t^{\frac{\pi}{2}} s\cos^2 s\,ds\right)dt\\
&=-8\int_0^{\frac{\pi}{2}}\left(\int_0^{s}\frac{1}{\cos^2(t)}\,dt\right) s\cos^2 s\,ds\\
&=-8\int_0^{\frac{\pi}{2}} s\cos s\sin s\,ds=-\pi.
\end{aligned}
$$

It is easy to see that the function q satisfies the following equations:

$$
\frac{1}{2}q''\cos t-2q'\sin t-2q\cos t=-4\sin t, \tag{8}
$$

and

$$
\frac{\cos^2 t}{2(1+\cos^2 t)}q'''-\frac{2\cos t\sin t}{1+\cos^2 t}q''-2q'=-\frac{4}{1+\cos^2 t}. \tag{9}
$$

The last equation implies that $q' = \xi''$ cannot achieve its non-positive local minimum at a point in $(-\pi/2, \pi/2)$. On the other hand, $\xi(\pm\pi/2) = 0$, and $\xi'(\pm\pi/2) = \pm 2\pi/3$. It is also easy to compute from equation (4) that $\xi''(\pm\pi/2) = 2$. Therefore $\xi''(t) > 0$ and ξ' is increasing on $[-\pi/2, \pi/2]$. Since $\xi'(0) = 0$, we have $\xi'(t) < 0$ on $(-\pi/2, 0)$ and $\xi'(t) > 0$ on $(0, \pi/2)$.

q'' satisfies the equation

$$\begin{aligned}(10)\qquad \frac{\cos^2 t}{2(1+\cos^2 t)}(q'')'' - \frac{\cos t \sin t(3 + 2\cos^2 t)}{(1+\cos^2 t)^2}(q'')' \\ - \frac{2(5\cos^2 t + \cos^4 t)}{(1+\cos^2 t)^2} q'' = -\frac{8\cos t \sin t}{(1+\cos^2 t)^2}.\end{aligned}$$

Thus q'' cannot achieve its non-positive local minimum at a point $t_1 \in (0, \pi/2)$. Otherwise, the left-hand side of the equation is non-negative at t_1 and the right hand side of the equation at t_1 is negative, which is impossible. Now $\xi'''(0) = q''(0) = 0$ since ξ is an even function and $\xi''' = q''$ is an odd function, and $\xi'''(\pi/2) = q''(\pi/2) = 8\pi/15$ since by $q(\pi/2) = \xi'(\pi/2) = 2\pi/3$, $q'(\pi/2) = \xi''(\pi/2) = 2$ and by (8),

$$q''\left(\frac{\pi}{2}\right) = \lim_{t\to\pi/2}\Big(2q'(t)\tan t + 2q(t) - 4\tan t\Big) = -2q''\left(\frac{\pi}{2}\right) + \frac{4\pi}{3}.$$

Therefore $\xi'''(t) > 0$ on $(0, \pi/2)$. The remaining results on the last line of the theorem follow from the fact that ξ''' is an odd function.

Set $h(t) = \xi''(t)t - \xi'(t)$. Then $h(0) = 0$ and $h'(t) = \xi'''(t)t > 0$ in $(0, \pi/2)$. Therefore $(\xi'(t)/t)' = h(t)/t^2 > 0$ in $(0, \pi/2)$. Note that $\xi'(-t)/(-t) = \xi'(t)/t$, $\lim_{t\to 0} \xi'(t)/t = \xi''(0) = 2(3 - \pi^2/4)$ and $\xi'(t)/t|_{t=\pi/2} = 4/3$. $\square$

3. An estimate on the first non-zero eigenvalue

We now estimate the first non-zero eigenvalue, using the method in [5]. Let v be an eigenfunction of the first non-zero eigenvalue λ of the Laplacian Δ

$$(11)\qquad \Delta v = -\lambda v \quad \text{in } M.$$

We may scale v such that

$$(12)\qquad \max_M v = 1, \quad \min_M v = -k$$

with $0 < k \le 1$.

Let $b > 1$ be an arbitrary constant and let

$$(13)\qquad \alpha = \frac{1}{2}(n-1)\kappa < 0 \qquad \text{and} \qquad \delta = \frac{\alpha}{\lambda} < 0.$$

Define a function Z on $[-\sin^{-1}(k/b), \sin^{-1}(1/b)]$ by

$$Z(t) = \max_{x\in M, t=\sin^{-1}(v(x)/b)} \frac{|\nabla v|^2}{\lambda(b^2 - v^2)}.$$

Theorem 3.1. *Suppose the function* $z: [-\sin^{-1}(k/b), \sin^{-1}(1/b)] \mapsto \mathbf{R}^1$ *satisfies the following conditions:*

(i) $z(t) \geq Z(t), \quad t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)]$.

(ii) *There exists some* $x_0 \in M$ *such that* $z(t_0) = Z(t_0)$ *at the point* $t_0 = \sin^{-1}\big(v(x_0)/b\big)$.

(iii) $z(t_0) \geq 1$.

(iv) $z'(t_0)\sin t_0 \leq 0$.

Then the following inequality holds:

$$0 \geq -\frac{1}{2}z''(t_0)\cos^2 t_0 + z'(t_0)\cos t_0 \sin t_0 + z(t_0) - 1 + 2\delta\cos^2 t_0. \tag{14}$$

Proof. Define

$$J(x) = \left\{\frac{|\nabla v|^2}{(b^2 - v^2)} - \lambda z\right\}\cos^2 t,$$

where $t = \sin^{-1}(v(x)/b)$. Then

$$J(x) \leq 0 \quad \text{for } x \in M \qquad \text{and} \qquad J(x_0) = 0.$$

If $\nabla v(x_0) = 0$, then

$$0 = J(x_0) = -\lambda z \cos^2 t.$$

This contradicts condition (iii) in the theorem. Therefore

$$\nabla v(x_0) \neq 0.$$

The maximum principle implies that

$$\nabla J(x_0) = 0 \qquad \text{and} \qquad \Delta J(x_0) \leq 0. \tag{15}$$

$J(x)$ can be rewritten as

$$J(x) = \frac{1}{b^2}|\nabla v|^2 - \lambda z \cos^2 t.$$

Take normal coordinates about x_0. (15) is equivalent to

$$(2/b^2)\sum_i v_i v_{ij}\big|_{x_0} = \lambda \cos t[z'\cos t - 2z\sin t]t_j\big|_{x_0} \tag{16}$$

and

$$\begin{aligned} 0 \geq\ & (2/b^2)\sum_{i,j} v_{ij}^2 + (2/b^2)\sum_{i,j} v_i v_{ijj} - \lambda(z''|\nabla t|^2 + z'\Delta t)\cos^2 t \\ & + 4\lambda z'\cos t \sin t|\nabla t|^2 - \lambda z \Delta \cos^2 t\big|_{x_0}. \end{aligned} \tag{17}$$

Rotate the frame so that $v_1(x_0) \neq 0$ and $v_i(x_0) = 0$ for $i \geq 2$. Then (16) implies

$$v_{11}\big|_{x_0} = (\lambda b/2)(z'\cos t - 2z\sin t)\big|_{x_0} \quad \text{and} \quad v_{1i}\big|_{x_0} = 0 \text{ for } i \geq 2. \tag{18}$$

It is easy to verify that the following equations:

$$\begin{aligned}
|\nabla v|^2\big|_{x_0} &= \lambda b^2 z\cos^2 t\big|_{x_0}, \qquad |\nabla t|^2\big|_{x_0} = \frac{|\nabla v|^2}{(b^2-v^2)} = \lambda z\big|_{x_0},\\
\Delta v/b\big|_{x_0} &= \Delta \sin t = \cos t\Delta t - \sin t|\nabla t|^2\big|_{x_0},\\
\Delta t\big|_{x_0} &= \frac{\sin t|\nabla t|^2 + \Delta v/b}{\cos t} = \frac{\lambda z\sin t - \lambda v/b}{\cos t}\Big|_{x_0},\\
\Delta \cos^2 t\big|_{x_0} &= \Delta\left(1 - v^2/b^2\right) = -(2/b^2)|\nabla v|^2 - (2/b^2)v\Delta v\\
&= -2\lambda z\cos^2 t + (2/b^2)\lambda v^2\big|_{x_0}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(2/b^2)\sum_{i,j} v_{ij}^2\big|_{x_0} &\geq (2/b^2)v_{11}^2\\
&= (1/2)\lambda^2(z')^2\cos^2 t - 2\lambda^2 zz'\cos t\sin t + 2\lambda^2 z^2\sin^2 t\big|_{x_0},\\
(2/b^2)\sum_{i,j} v_i v_{ijj}\big|_{x_0} &= (2/b^2)\left(\nabla v\,\nabla(\Delta v) + \operatorname{Ric}(\nabla v,\nabla v)\right)\\
&\geq (2/b^2)(\nabla v\,\nabla(\Delta v) + (n-1)\kappa|\nabla v|^2)\\
&= -2\lambda^2 z\cos^2 t + 4\alpha\lambda z\cos^2 t\big|_{x_0}\\
&\qquad - \lambda(z''|\nabla t|^2 + z'\Delta t)\cos^2 t\big|_{x_0}\\
&= -\lambda^2 zz''\cos^2 t - \lambda^2 zz'\cos t\sin t + (1/b)\lambda^2 z' v\cos t\big|_{x_0},
\end{aligned}$$

and

$$\begin{aligned}
&4\lambda z'\cos t\sin t|\nabla t|^2 - \lambda z\Delta\cos^2 t\big|_{x_0}\\
&\qquad = 4\lambda^2 zz'\cos t\sin t + 2\lambda^2 z^2\cos^2 t - (2/b)\lambda^2 zv\sin t\big|_{x_0}.
\end{aligned}$$

Putting these results into (17) we get

$$\begin{aligned}
0 \geq\; & -\lambda^2 zz''\cos^2 t + (1/2)\lambda^2(z')^2\cos^2 t + \lambda^2 z'\cos t\,(z\sin t + \sin t)\\
& + 2\lambda^2 z^2 - 2\lambda^2 z + 4\alpha\lambda z\cos^2 t\big|_{x_0},
\end{aligned} \tag{19}$$

where we used (18). By condition (iii) in the theorem,

$$z(t_0) \geq 1. \tag{20}$$

Dividing both sides of (19) by $2\lambda^2 z\big|_{x_0}$, we have

$$\begin{aligned}
0 \geq\; & -\frac{1}{2}z''(t_0)\cos^2 t_0 + z'(t_0)\cos t_0\sin t_0 + z(t_0) - 1 + 2\delta\cos^2 t_0\\
& + \frac{1}{2}z'(t_0)\sin t_0\cos t_0\left(\frac{1}{z(t_0)} - 1\right) + \frac{1}{4z(t_0)}(z'(t_0))^2\cos^2 t_0.
\end{aligned} \tag{21}$$

By conditions (iii) and (iv) in the theorem, the last two terms are nonnegative. Therefore (14) follows. $\square$

Proof of Theorem 1.1. Let

$$z(t) = 1 + \delta\xi(t), \tag{22}$$

where ξ is the function defined by (3) in Theorem 2.1 and δ is the negative constant in (13). We claim that

$$Z(t) \le z(t) \qquad \text{on } [-\sin^{-1}(k/b), \sin^{-1}(1/b)]. \tag{23}$$

Theorem 2.1 implies that for $t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)]$, we have the following:

$$\begin{aligned} &\frac{1}{2}z'' \cos^2 t - z' \cos t \sin t - z = -1 + 2\delta \cos^2 t, \\ &z'(t) \sin t \le 0 \quad (\text{since } \delta < 0), \quad \text{and} \\ &z(t) \ge z\left(\frac{\pi}{2}\right) = 1. \end{aligned}$$

Let $P \in \mathbf{R}^1$ and $t_0 \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)]$ be such that

$$P = \max_{t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)]} (Z(t) - z(t)) = Z(t_0) - z(t_0).$$

Thus

$$\begin{aligned} Z(t) &\le z(t) + P \quad \text{on } [-\sin^{-1}(k/b), \sin^{-1}(1/b)] \quad \text{and} \\ Z(t_0) &= z(t_0) + P. \end{aligned} \tag{24}$$

Suppose that $P > 0$. Then $z + P$ satisfies the conditions in Theorem 3.1. (14) implies that

$$\begin{aligned} z(t_0) + P &= Z(t_0) \\ &\le \frac{1}{2}(z+P)''(t_0) \cos^2 t_0 - (z+P)'(t_0) \cos t_0 \sin t_0 + 1 - 2\delta \cos^2 t_0 \\ &= \frac{1}{2}z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 + 1 - 2\delta \cos^2 t_0 \\ &= z(t_0). \end{aligned}$$

This contradicts the assumption $P > 0$. Thus $P \le 0$ and (23) must hold. This means

$$\sqrt{\lambda} \ge \frac{|\nabla t|}{\sqrt{z(t)}}. \tag{25}$$

Note that the eigenfunction v of the first non-zero eigenvalue has exactly two nodal domains $D^+ = \{x : v(x) > 0\}$ and $D^- = \{x : v(x) < 0\}$ (cf. [1]) and that the nodal set $v^{-1}(0)$ is compact. Take q_1 on M such that $v(q_1) = 1 = \sup_M v$ and $q_2 \in v^{-1}(0)$ such that distance $d(q_1, q_2) =$ distance $d(q_1, v^{-1}(0))$. Let L be the minimal geodesic segment between q_1 and q_2. We integrate both sides of (25) along L and change variables, and let $b \to 1$. Let r be the larger of the two interior radii of the nodal domains. Then

$$
\begin{aligned}
(26) \qquad r\sqrt{\lambda} &\geq \int_L \frac{|\nabla t|}{\sqrt{z(t)}}\, dl \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{z(t)}}\, dt \geq \frac{\left(\int_0^{\pi/2} \; dt\right)^{\frac{3}{2}}}{\left(\int_0^{\pi/2} \; z(t)\, dt\right)^{\frac{1}{2}}} \geq \left(\frac{(\pi/2)^3}{\int_0^{\pi/2} \; z(t)\, dt}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Squaring the two sides, we get

$$\lambda \geq \frac{\pi^3}{8r^2 \int_0^{\pi/2} \; z(t)\, dt}.$$

Now

$$\int_0^{\frac{\pi}{2}} \; z(t)\, dt = \int_0^{\frac{\pi}{2}} \; [1 + \delta\xi(t)]\, dt = \frac{\pi}{2}(1-\delta),$$

by (6) in Theorem 2.1. Therefore

$$\lambda \geq \frac{\pi^2}{4r^2(1-\delta)} \quad \text{and} \quad \lambda \geq \frac{\pi^2}{4r^2} + \frac{1}{2}(n-1)\kappa. \qquad \square$$

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