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LIFTING A GENERIC MAP OF A SURFACE INTO THE PLANE TO AN EMBEDDING INTO 4-SPACE

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Dedicated to Professor Takao Matumoto in celebration of his 60th birthday.

Abstract. Let $f : M \to \mathbf{R}^2$ be a stable map of a closed surface M into the plane and $\pi_2^2 : \mathbf{R}^4 \to \mathbf{R}^2$ the orthogonal projection. In this paper, we will show that for any such f there exists an embedding $F : M \to \mathbf{R}^4$ such that $f = \pi_2^2 \circ F$ is satisfied.

1. Introduction

Throughout the paper, all manifolds and maps are differentiable of class C^∞ and $\pi_p^k : \mathbf{R}^p \times \mathbf{R}^k \to \mathbf{R}^p$ is the orthogonal projection onto the first factor.

Let M be a closed surface and $F : M \to \mathbf{R}^4$ an embedding. If we take a generic projection $\tilde{\pi} : \mathbf{R}^4 \to \mathbf{R}^2$, then the composition $\tilde{\pi} \circ F$ is a stable map (see Mather [15]). Here, a stable map means that any small perturbation of this stable map can be obtained from it by composition with diffeomorphisms of the source and target manifolds. For the precise definition, see Subsection 2.1.

Conversely, we prove the following theorem in this paper.

Theorem 1.1. *Let M be a closed surface and $\pi_2^2 : \mathbf{R}^4 \to \mathbf{R}^2$ the orthogonal projection. For any stable map $f : M \to \mathbf{R}^2$, there exists an embedding $F : M \to \mathbf{R}^4$ such that $f = \pi_2^2 \circ F$ is satisfied.*

Let M be a closed n-dimensional manifold and $f : M \to \mathbf{R}^p$ a stable map. If there exists an embedding $F : M \to \mathbf{R}^{p+k}$ such that $\pi_p^k \circ F = f$, we call such an F an *embedding lift* of f. Therefore, Theorem 1.1 is a result about the existence of an embedding lift.

We have the following known results about the existence of an embedding lift. In the case where M is a closed surface, Giller [9] gave a criterion for lifting a generic immersion $f : M \looparrowright \mathbf{R}^3$ to an embedding in $\mathbf{R}^4$. For the same

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setting, Akhmetiev [2] and Carter–Saito [7] independently provided a criterion when f is a generic map and M is an oriented surface. Regardless of the orientability of M, Carter–Saito and Satoh [7], [19] obtained several necessary and sufficient conditions. In the case where M is a closed n-dimensional manifold greater than one, Saeki and Sakuma [18] gave a necessary and sufficient condition for lifting a stable map without triple points $f : M \to \mathbf{R}^{2n-1}$ to an embedding in $\mathbf{R}^{2n}$. Note that in the case where M is a closed surface, Carrara, Ruas and Saeki [5] studied a stable map $f : M \to \mathbf{R}^2$ which has the standard lifting property in $\mathbf{R}^4$. Let $f : M \to N$ be a (continuous) map between n-dimensional manifolds, where M is compact and N is stably parallelizable. We say that f is realizable in $\mathbf{R}^{2n}$ if the composition of f and some embedding $i : N \to \mathbf{R}^{2n}$ is C^0-close to an embedding. Akhmetiev [1], [3] studied the problem of realizing a map $f : S^n \to S^n$ in $\mathbf{R}^{2n}$ for $n > 2$. Melikhov [16] gave a necessary and sufficient condition for realizing a map $f : M \to N$ in $\mathbf{R}^{2n}$ $(n > 2)$.

The paper is organized as follows. In Section 2, we give the definition of a stable map and prepare some tools for the proof of Theorem 1.1. In Section 3, we prove Theorem 1.1. In Section 4, we give two examples which clarify Theorem 1.1 and consider the relationship between the results obtained in [7], [9], [18], [19] and Theorem 1.1.

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2. Preliminaries

2.1. Definition of a stable map. Let $f : M \to \mathbf{R}^p$ be a smooth map of a closed n-dimensional manifold M into $\mathbf{R}^p$. We denote the set of such maps by $C^\infty(M, \mathbf{R}^p)$, which is equipped with the Whitney C^∞-topology. A smooth map f is said to be a *stable map* if in $C^\infty(M, \mathbf{R}^p)$ there exists an open neighborhood U of f such that for any $g \in U$, g is C^∞ right-left equivalent to f, i.e., there exist two diffeomorphisms $\Phi : M \to M$ and $\varphi : \mathbf{R}^p \to \mathbf{R}^p$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\Phi"] \arrow[d, "f"'] & M \arrow[d, "g"] \\ \mathbf{R}^p \arrow[r, "\varphi"'] & \mathbf{R}^p \end{tikzcd}$$

is commutative.

For a smooth map $f : M \to \mathbf{R}^p$, we denote by $S(f)$ the set of the points in M where the rank of the differential of f is strictly less than $\min(n, p)$. We say that a point $q \in S(f)$ is a *singular point* of f.

In the cases where $p = 1$ or $(n, p) = (2, 2)$, the following characterizations of stable maps are well-known (see [10], [20], for example).

PROPOSITION 2.1. *A smooth function $f : M \to \mathbf{R}^1$ is a stable map if and only if the following conditions are satisfied.*

(i) *For every $q \in M$, there exist local coordinates $(x_1, \dots, x_n)$ and X around $q \in M$ and $f(q) \in \mathbf{R}^1$, respectively, such that one of the following holds*:
   (a) $X \circ f = x_1$ (*q is a regular point*),
   (b) $X \circ f = \pm x_1^2 \pm \cdots \pm x_n^2$ (*q is a singular point*).

(ii) *For any two distinct singular points q_1 and q_2 of f, $f(q_1) \neq f(q_2)$ is satisfied.*

We call such a stable map $f : M \to \mathbf{R}^1$ a *stable Morse function*.

PROPOSITION 2.2. *A smooth map $f : M \to \mathbf{R}^2$ of a closed surface M is a stable map if and only if the following conditions are satisfied.*

(i) *For every $q \in M$, there exist local coordinates (x, y) and (X, Y) around $q \in M$ and $f(q) \in \mathbf{R}^2$, respectively, such that one of the following holds*:
   (a) $(X \circ f, Y \circ f) = (x, y)$ (*q is a regular point*),
   (b) $(X \circ f, Y \circ f) = (x, y^2)$ (*q is a fold point*),
   (c) $(X \circ f, Y \circ f) = (x, xy - y^3)$ (*q is a cusp point*).

(ii) *If $q \in M$ is a cusp point, then $f^{-1}(f(q)) \cap S(f) = \{q\}$.*

(iii) *The map $f|(S(f) \setminus \{\text{cusp points}\})$ is an immersion with normal crossings.*

For a stable map $f : M \to \mathbf{R}^2$ of a closed surface M, we denote by $C(f) \subset M$ the set of all cusp points in M and by $N(f) \subset \mathbf{R}^2$ the set of all normal crossing points of $f(S(f))$. Note that $S(f)$ is a compact 1-dimensional submanifold of M. Both $C(f)$ and $N(f)$ have a finite number of elements.

REMARK 2.3. Let $f : M \to \mathbf{R}^2$ be a stable map of a closed surface. By the image of singular points $f(S(f)) \subset \mathbf{R}^2$, $\mathbf{R}^2$ is naturally stratified into 2-, 1- and 0-dimensional strata. Note that the union of 1- and 0-dimensional strata forms $f(S(f))$ and the union of 0-dimensional strata corresponds to $f(C(f)) \cup N(f)$. On each 1-dimensional stratum of $f(S(f))$, we can define an orientation as follows. We fix the canonical orientation on $\mathbf{R}^2$. Let Ω be a connected component of $\mathbf{R}^2 \setminus f(S(f))$. We associate to Ω a non-negative integer $n_f(\Omega)$, which is the number of points in the fiber of f over any point of Ω. Every 1-dimensional stratum in $f(S(f))$ is adjacent to exactly two connected components of $\mathbf{R}^2 \setminus f(S(f))$. Since these two components have distinct $n_f(\Omega)$-values, we can orient each 1-dimensional stratum in $f(S(f))$ so that the region with the larger $n_f(\Omega)$-value is on its left.

2.2. Projection of a stable map. Let $f : M \to \mathbf{R}^2$ be a stable map of a closed surface M and $\pi : \mathbf{R}^2 \to \mathbf{R}^1$ a generic projection such that $\pi \circ f$ is a stable Morse function. There always exists such a generic projection π for any stable map f (see [4], [6], for example).

Definition 2.4. We say that a point $r \in f(S(\pi \circ f)) \cup f(C(f)) \cup N(f)$ is a *bifurcation point* of $\{\pi, f\}$. We call $t \in \mathbf{R}^1$ a *bifurcation value* of $\{\pi, f\}$ if $\pi^{-1}(t)$ contains a bifurcation point, otherwise we call $t \in \mathbf{R}^1$ a *non-bifurcation value* of $\{\pi, f\}$.

Note that the number of bifurcation points of $\{\pi, f\}$ is finite and we may assume that for each bifurcation value t of $\{\pi, f\}$, there exists exactly one bifurcation point in $\pi^{-1}(t)$.

In the following, we study the behavior of $f(S(f))$ with the generic projection π. Mancini and Ruas [14] determined local forms of π and f. See also [6], [8]. Let $r \in f(S(f)) \setminus (f(S(\pi \circ f)) \cup f(C(f)) \cup N(f))$ be an image of singular point of f, but not a bifurcation point of $\{\pi, f\}$, and $q \in M$ the unique singular point of f such that $r = f(q)$. By taking suitable local coordinates around $q \in M$, $r \in \mathbf{R}^2$ and $\pi(r) \in \mathbf{R}^1$, π and f can be expressed by the following form:

$$(x, y) \overset{f}{\mapsto} (x, y^2) \overset{\pi}{\mapsto} x \qquad \text{(see Figure 1(a)).} \tag{2.1}$$

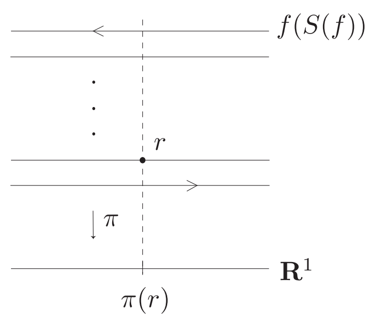

Figure 1. (a)

Let $r \in f(S(\pi \circ f)) \cup f(C(f))$ be a bifurcation point of $\{\pi, f\}$ and $q \in M$ the singular point of $\pi \circ f$ or the cusp point of f such that $r = f(q)$. By taking suitable local coordinates around $q \in M$, $r \in \mathbf{R}^2$ and $\pi(r) \in \mathbf{R}^1$, π and f can be expressed by one of the following forms:

$$(x, y) \overset{f}{\mapsto} (x^2 + y^2, y) \overset{\pi}{\mapsto} x^2 + y^2 \qquad \text{(see Figure 1(b))}, \tag{2.2}$$

$$(x, y) \overset{f}{\mapsto} (-x^2 + y^2, y) \overset{\pi}{\mapsto} -x^2 + y^2 \qquad \text{(see Figure 1(c))}, \tag{2.3}$$

$$(x, y) \overset{f}{\mapsto} (x, y^3 - xy) \overset{\pi}{\mapsto} x \qquad \text{(see Figure 1(d))}. \tag{2.4}$$

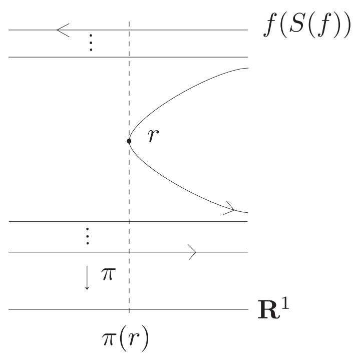

FIGURE 1. (b)

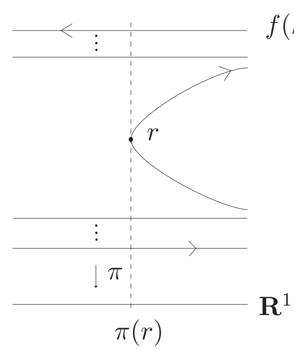

FIGURE 1. (c)

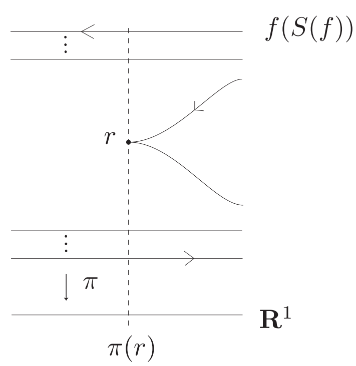

FIGURE 1. (d)

Let $r \in N(f)$ be a bifurcation point of $\{\pi, f\}$ and $q_i \in M$ the fold points of f such that $r = f(q_i)$ $(i = 1, 2)$ and $q_1 \neq q_2$. By taking suitable local coordinates around $q_1, q_2 \in M$, $r \in \mathbf{R}^2$ and $\pi(r) \in \mathbf{R}^1$, π and f can be expressed by one of the following forms:

$$\begin{aligned} &(x_1, y_1) \overset{f}{\mapsto} (x_1, y_1^2 + x_1) \overset{\pi}{\mapsto} x_1, \\ &(x_2, y_2) \overset{f}{\mapsto} (x_2, y_2^2 - x_2) \overset{\pi}{\mapsto} x_2 \qquad \text{(see Figure 1(e))}, \end{aligned} \tag{2.5}$$

$$(2.6)\qquad \begin{aligned}&(x_1, y_1) \overset{f}{\mapsto} (x_1, y_1^2 + x_1) \overset{\pi}{\mapsto} x_1,\\ &(x_2, y_2) \overset{f}{\mapsto} (x_2, -y_2^2 - x_2) \overset{\pi}{\mapsto} x_2 \qquad \text{(see Figure 1(f)).}\end{aligned}$$

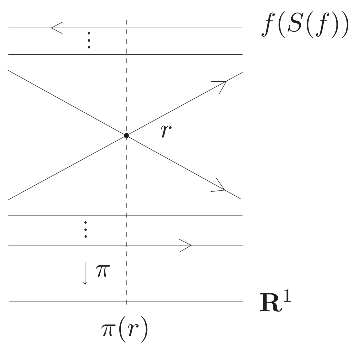

FIGURE 1. (e)

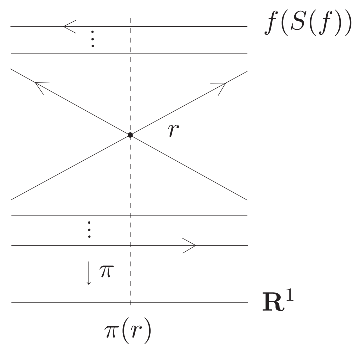

FIGURE 1. (f)

2.3. Graphs on cylinders. Let $f : M \to \mathbf{R}^2$ be a stable map of a closed surface M and $\pi : \mathbf{R}^2 \to \mathbf{R}^1$ a generic projection such that $\pi \circ f$ is a stable Morse function. We set $M_t = (\pi \circ f)^{-1}(t)$ for $t \in \mathbf{R}^1$. For any non-bifurcation value t of $\{\pi, f\}$, M_t is a closed 1-dimensional manifold (possibly disconnected or empty) and $f|M_t : M_t \to \pi^{-1}(t)$ is a stable Morse function.

Let $t \in \pi \circ f(M)$ be a non-bifurcation value of $\{\pi, f\}$. Note that M_t is a disjoint union of finitely many circle components, say $M_t = M_{t,1} \cup \cdots \cup M_{t,k}$ ($k \geq 1$). We let that $\mathbf{R}_1^2$ be the fiber of the orthogonal projection $\pi_2^2 : \mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}_1^2 \to \mathbf{R}^2$. Suppose that $D_{t,1}^2, \ldots, D_{t,k}^2$ are mutually disjoint 2-disks embedded in $\mathbf{R}_1^2$. Let us consider an embedding $F_t : M_t \to \pi^{-1}(t) \times \mathbf{R}_1^2$. If we have that $\pi_2^2 \circ F_t = f|M_t$ and that $(\mathrm{Pr} \circ F_t)|M_{t,i} : M_{t,i} \to \partial D_{t,i}^2$ is a diffeomorphism for each $M_{t,i}$, we call F_t a *graph* of $f|M_t : M_t \to \pi^{-1}(t)$. Here, $\mathrm{Pr} : \pi^{-1}(t) \times \mathbf{R}_1^2 \to \mathbf{R}_1^2$ is the projection onto the second factor. See Figure 2, for example. If $t \notin \pi \circ f(M)$, we consider that any map $F_t : M_t = \emptyset \to \pi^{-1}(t) \times \mathbf{R}_1^2$ is a graph of $f|M_t$.

3. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following lemma.

LEMMA 3.1. *Let $f : M \to \mathbf{R}^2$ be a stable map of a closed surface M and $\pi : \mathbf{R}^2 \to \mathbf{R}^1$ a generic projection such that $\pi \circ f$ is a stable Morse function. Let t_1 and t_2 be non-bifurcation values of $\{\pi, f\}$ such that $t_1 < t_2$. If there is an embedding $F_{t_1} : M_{t_1} \to \pi^{-1}(t_1) \times \mathbf{R}_1^2$ which is a graph of $f|M_{t_1} : M_{t_1} \to \pi^{-1}(t_1)$, then we can construct an embedding*

$$F_{[t_1,t_2]} : M_{[t_1,t_2]} \to \pi^{-1}([t_1, t_2]) \times \mathbf{R}_1^2$$

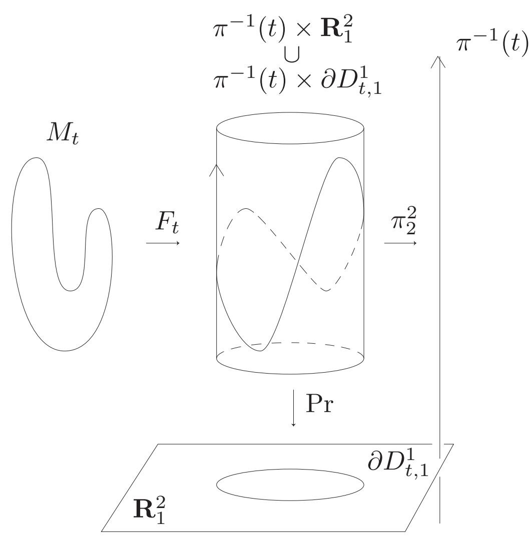

FIGURE 2

such that $F_{[t_1,t_2]}|M_{t_1} = F_{t_1}$, $\pi^2_2 \circ F_{[t_1,t_2]} = f|M_{[t_1,t_2]}$ *and* $F_{[t_1,t_2]}|M_{t_2}$ *is a graph of* $f|M_{t_2} : M_{t_2} \to \pi^{-1}(t_2)$*. Here, we define* $M_{[t_1,t_2]} = (\pi \circ f)^{-1}([t_1, t_2])$*.*

Since M is compact, there exists a closed interval $[\alpha, \beta] \subset \mathbf{R}^1$ such that $\pi \circ f(M) \subsetneq [\alpha, \beta]$. If we put $\alpha = t_1$ and $\beta = t_2$ in Lemma 3.1, an embedding $F_{[\alpha,\beta]} : M \to \mathbf{R}^4$ is a desired embedding lift of f. This completes the proof of Theorem 1.1. $\square$

Proof of Lemma 3.1. Suppose that the closed interval $[t_1, t_2]$ does not have a bifurcation value of $\{\pi, f\}$. For this case, π and f can be described as (2.1) (see Figure 1(a)) and it is easy to construct a required embedding $F_{[t_1,t_2]} : M_{[t_1,t_2]} \to \pi^{-1}([t_1, t_2]) \times \mathbf{R}^2_1$.

Suppose that the closed interval $[t_1, t_2]$ has bifurcation values of $\{\pi, f\}$. We may assume that $[t_1, t_2]$ has exactly one bifurcation value $b \in (t_1, t_2)$ of $\{\pi, f\}$. We let $r \in f(S(f))$ be the bifurcation point such that $\pi(r) = b$. Since each circle $F_{t_1}(M_{t_1,i})$ is on $\pi^{-1}(t_1) \times \partial D^2_{t_1,i}$ and all solid cylinders $\pi^{-1}(t_1) \times D^2_{t_1,i}$ are mutually disjoint, we may consider only connected components of M_b that have at least one singular point $q \in S(f)$ such that $f(q) = r$.

Let π and f be expressed as the local form (2.2). If the positive direction of $\mathbf{R}^1$ is left to right in Figure 1(b), then $M_{[t_1,t_2]}$ is obtained by attaching a 0-handle to an empty set. It is easy to construct a required embedding $F_{[t_1,t_2]} : M_{[t_1,t_2]} \to \pi^{-1}([t_1, t_2]) \times \mathbf{R}^2_1$. See Figures 3. If the positive direction of $\mathbf{R}^1$ is right to left in Figure 1(b), then $M_{[t_1,t_2]}$ is obtained by attaching a 2-handle to a circle M_{t_1}. Since F_{t_1} is a graph of $f|M_{t_1} : M_{t_1} \to \pi^{-1}(t_1)$, we can construct a required embedding $F_{[t_1,t_2]} : M_{[t_1,t_2]} \to \pi^{-1}([t_1,t_2]) \times \mathbf{R}_1^2$. That is, we change t_1 and t_2 in Figures 3 (see Remark 3.2).

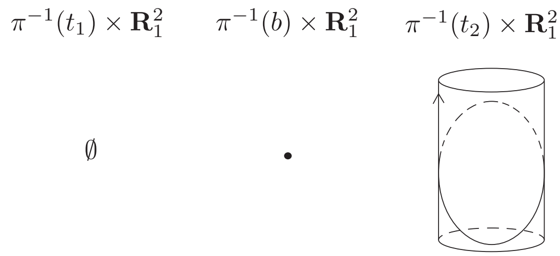

FIGURE 3

Let π and f be expressed as the local form (2.3). Let $q \in M_b \cap S(f)$ be the singular point of f such that $f(q) = r$. Then $M_{[t_1,t_2]}$ is obtained by attaching a 1-handle to M_{t_1}. Let $\varphi : J \times J \to \pi^{-1}(t_1) \times \mathbf{R}_1^2$ be an embedding such that $\varphi(J \times J) \cap F_{t_1}(M_{t_1}) = \varphi(\partial J \times J)$, where we set $J = [-1, 1]$. By this embedding φ, we have an embedding $F_{[t_1,t_2]} : M_{[t_1,t_2]} \to \pi^{-1}([t_1,t_2]) \times \mathbf{R}_1^2$. We say that $F_{[t_1,t_2]}$ and $F_{[t_1,t_2]}|M_{t_2}$ is obtained from F_{t_1} by a 1-handle operation along the 1-handle φ. We also call the arc $\varphi(J \times \{0\})$ the core of the 1-handle φ. If we can choose an orientation of M_{t_1} such that φ is consistent with this orientation (i.e., $\varphi(\partial(J \times J))$ is oriented and the inclusion $\varphi(\partial J \times J) \subset M_{t_1}$ is orientation reversing), we call the above operation an oriented 1-handle operation. Otherwise, we call it a non-oriented 1-handle operation. To prove Lemma 3.1, we have to perform a 1-handle operation so that $\pi_2^2 \circ F_{[t_1,t_2]} = f|M_{[t_1,t_2]}$ and $F_{[t_1,t_2]}|M_{t_2}$ is a graph of $f|M_{t_2} : M_{t_2} \to \pi^{-1}(t_2)$.

If the positive direction of $\mathbf{R}^1$ is left to right in Figure 1(c), both arcs $\varphi(\partial J \times J)$ of a 1-handle φ are at regular points of $f|M_{t_1} : M_{t_1} \to \pi^{-1}(t_1)$. Suppose that M_{t_1} is connected and a 1-handle φ is an oriented operation. In this case, we attach φ to M_{t_1} such that $\varphi(J \times J) \subset \pi^{-1}(t_1) \times D_{t_1}^2$. See Figure 4(a). In this figure, we depict the cylinder $\pi^{-1}(t_1) \times \partial D_{t_1}^2$ from the top $\{\infty\} \times \mathbf{R}_1^2$ to the bottom $\{-\infty\} \times \mathbf{R}_1^2$ and the black dots are the critical points of $f|M_{t_1} : M_{t_1} \to \pi^{-1}(t_1)$.

By this operation along φ, we have an embedding

$$F_{[t_1,t_2]} : M_{[t_1,t_2]} \to \pi^{-1}([t_1,t_2]) \times \mathbf{R}_1^2$$

such that $F_{[t_1,t_2]}|M_{t_1} = F_{t_1}$ and $\pi_2^2 \circ F_{[t_1,t_2]} = f|M_{[t_1,t_2]}$. Note that M_{t_2} has two components and each component $M_{t_2,i}$ has a new born critical point c_i of $f|M_{t_2}$, $i = 1, 2$. We can check that the embedding $F_{[t_1,t_2]}|M_{t_2}$ is a graph of $f|M_{t_2}$. Therefore, the embedding $F_{[t_1,t_2]}$ is a required one. See Figure 4(b). In this figure, we see each cylinder from the side and the top. Both $\mathfrak{A}$ and $\mathfrak{B}$ are parts of $M_{t_1} \setminus \varphi(\partial J \times J)$.

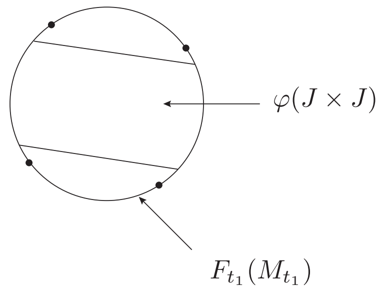

FIGURE 4. (a)

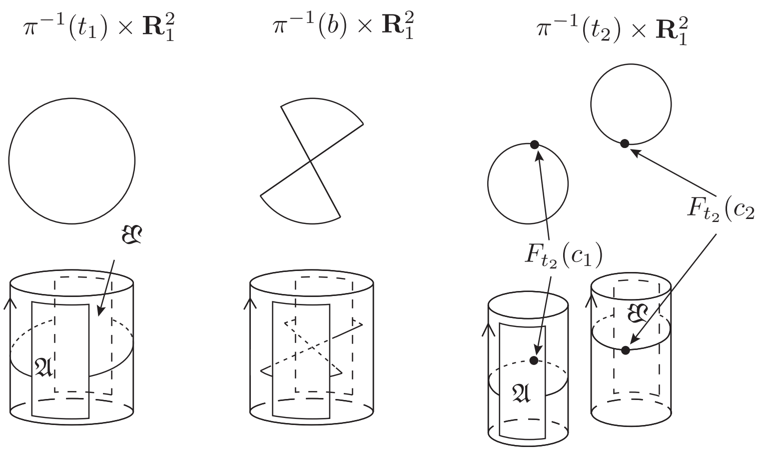

FIGURE 4. (b)

Suppose that M_{t_1} is connected and a 1-handle φ is a non-oriented operation. Let $c_0 \in M_{t_1}$ be the minimal critical point of $f|M_{t_1} : M_{t_1} \to \pi^{-1}(t_1)$ and $l(c_0) = \mathrm{Pr}^{-1}(\mathrm{Pr} \circ F_{t_1}(c_0))$ the line in $\pi^{-1}(t_1) \times \partial D_{t_1}$ passing through $F_{t_1}(c_0)$. In this case, we attach φ to M_{t_1} such that $\varphi([-1,0] \times J) \subset \pi^{-1}(t_1) \times D^2_{t_1}$, $\varphi([0,1] \times J) \subset \pi^{-1}(t_1) \times (\mathbf{R}^2_1 \setminus \mathrm{Int}\, D^2_{t_1})$ and $\varphi(\{0\} \times J) \subset l(c_0)$. See Figure 5(a). This figure has the same setting as Figure 4(a).

Let $\varepsilon \in \mathbf{R}^1$ be a sufficiently small positive number such that we have $b < b+\varepsilon < t_2$. After we perform the operation along φ, we have an embedding $F_{[t_1,b+\varepsilon]} : M_{[t_1,b+\varepsilon]} \to \pi^{-1}([t_1, b+\varepsilon]) \times \mathbf{R}^2_1$ such that $F_{[t_1,t_2]}|M_{t_1} = F_{t_1}$ and $\pi^2_2 \circ F_{[t_1,b+\varepsilon]} = f|M_{[t_1,b+\varepsilon]}$. Note that $M_{b+\varepsilon}$ is connected and $M_{b+\varepsilon}$ has two new born critical points c_i of $f|M_{b+\varepsilon}$, $i = 1, 2$. See Figure 5(b). In this figure, $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are parts of $M_{t_1} \setminus \varphi(\partial J \times J)$.

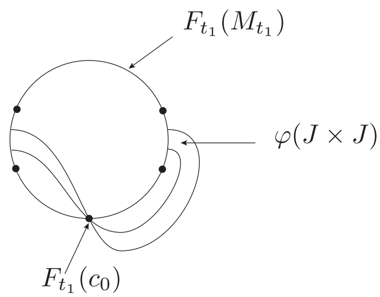

FIGURE 5. (a)

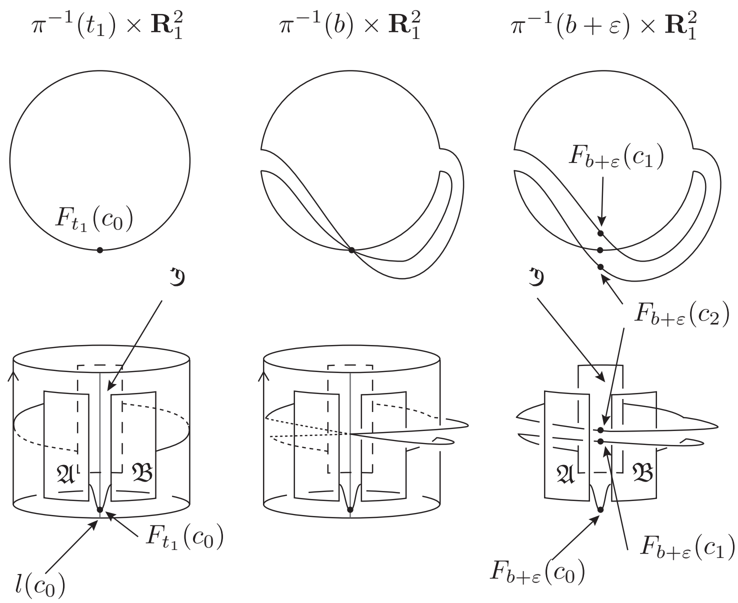

FIGURE 5. (b)

Since c_0 is also the minimal critical point of $f|M_{b+\varepsilon}$, we have an isotopy $\widetilde{F}_t : M_t \to \pi^{-1}(t) \times \mathbf{R}_1^2$ ($t \in [b+\varepsilon, t_2]$) such that $\widetilde{F}_{b+\varepsilon} = F_{[t_1,b+\varepsilon]}|M_{b+\varepsilon}$, $\pi_2^2 \circ \widetilde{F}_t = f|M_t$ and $\widetilde{F}_{t_2}$ is a graph of $f|M_{t_2} : M_{t_2} \to \pi^{-1}(t_2)$. See Figure 5(c). By gluing the embedding $F_{[t_1,b+\varepsilon]}$ and the isotopy $\widetilde{F}_t$, we have a desired embedding $F_{[t_1,t_2]} : M_{[t_1,t_2]} \to \pi^{-1}([t_1,t_2]) \times \mathbf{R}_1^2$.

Suppose that M_{t_1} has two components. In this case, a 1-handle operation φ is always an oriented operation and we attach φ to M_{t_1} such that $\varphi(J \times J) \subset$

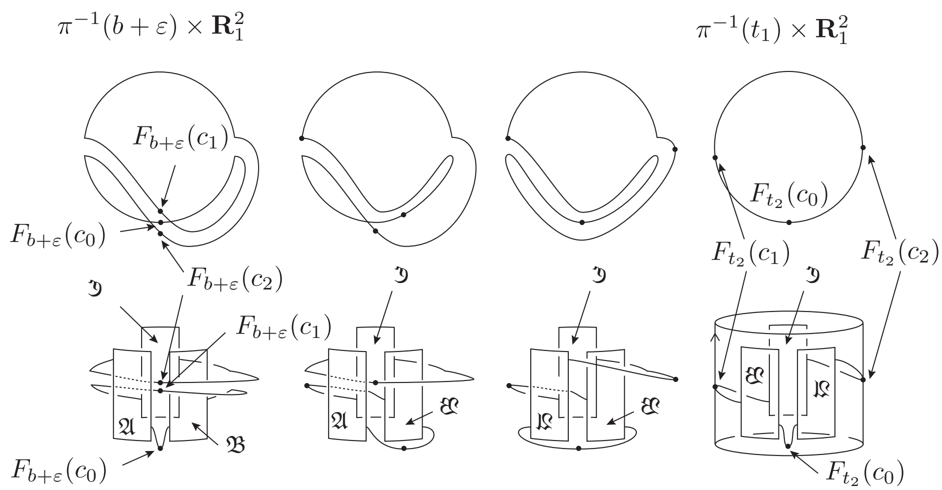

FIGURE 5. (c)

$\pi^{-1}(t_1) \times (\mathbf{R}_1^2 \setminus (\mathrm{Int}\, D^2_{t_1,1} \cup \mathrm{Int}\, D^2_{t_1,2}))$. See Figure 6(a). This figure has the same setting as Figure 4(a).

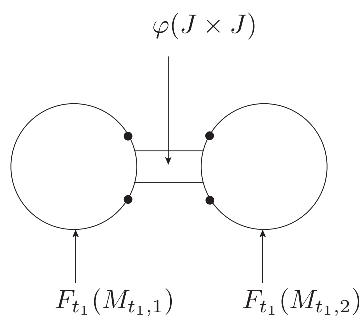

FIGURE 6. (a)

From this operation along φ, we have an embedding

$$F_{[t_1,t_2]} : M_{[t_1,t_2]} \to \pi^{-1}([t_1,t_2]) \times \mathbf{R}_1^2$$

such that $F_{[t_1,t_2]}|M_{t_1} = F_{t_1}$ and $\pi_2^2 \circ F_{[t_1,t_2]} = f|M_{[t_1,t_2]}$. Note that M_{t_2} is connected and M_{t_2} has two new born critical points c_i of $f|M_{t_2}$, $i = 1, 2$. We can check that the embedding $F_{[t_1,t_2]}|M_{t_2}$ is a graph of $f|M_{t_2}$. See Figure 6(b). In this figure, $\mathfrak{A}$ and $\mathfrak{B}$ are parts of $M_{t_1} \setminus \varphi(J \times J)$.

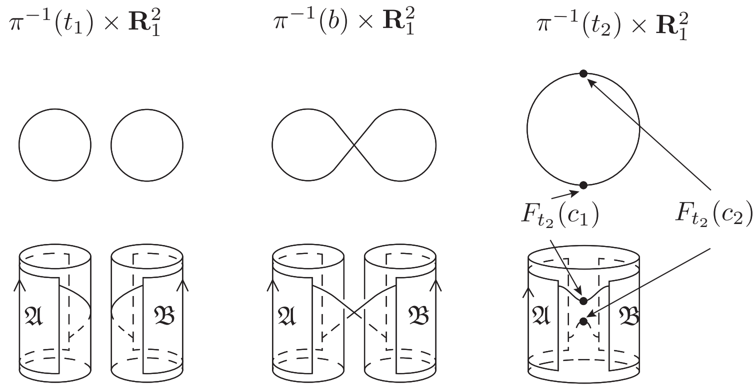

FIGURE 6. (b)

If the positive direction of $\mathbf{R}^1$ is right to left in Figure 1(c), the core of a 1-handle φ connects two critical points c_1 and $c_2 \in M_{t_1}$ of $f|M_{t_1} : M_{t_1} \to \pi^{-1}(t_1)$ which are eliminated by $f|M_{[t_1,t_2]}$. Suppose that M_{t_1} is connected and a 1-handle φ is an oriented operation. In this case, we attach φ to M_{t_1} such that $\varphi(J \times J) \subset \pi^{-1}(t_1) \times D^2_{t_1}$. See Figure 7(a). Suppose that M_{t_1} is connected and a 1-handle φ is a non-oriented operation. In this case, we attach φ to M_{t_1} such that $\varphi([-1,0] \times J) \subset \pi^{-1}(t_1) \times D^2_{t_1}$, $\varphi([0,1] \times J) \subset \pi^{-1}(t_1) \times (\mathbf{R}_1^2 \setminus \operatorname{Int} D^2_{t_1})$ and $\varphi(\{0\} \times J) \subset l(c_0)$. Here, $c_0 \in M_{t_1}$ is the minimal critical point of $f|M_{t_1} : M_{t_1} \to \pi^{-1}(t_1)$ and $l(c_0) = \operatorname{Pr}^{-1}(\operatorname{Pr} \circ F_{t_1}(c_0))$ is the line in $\pi^{-1}(t_1) \times \partial D_{t_1}$ passing through $F_{t_1}(c_0)$. See Figure 7(b). Suppose that M_{t_1} has two components. In this case, a 1-handle operation φ is always an oriented operation and we attach φ to M_{t_1} such that $\varphi(J \times J) \subset \pi^{-1}(t_1) \times (\mathbf{R}_1^2 \setminus (\operatorname{Int} D^2_{t_1,1} \cup \operatorname{Int} D^2_{t_1,2}))$. See Figure 7(c).

From each operation along φ, we have an embedding

$$F_{[t_1,t_2]} : M_{[t_1,t_2]} \to \pi^{-1}([t_1,t_2]) \times \mathbf{R}_1^2,$$

which is a required one. We leave it to the reader to check that we have $F_{[t_1,t_2]}|M_{t_1} = F_{t_1}$ and $\pi_2^2 \circ F_{[t_1,t_2]} = f|M_{[t_1,t_2]}$, and that the embedding $F_{[t_1,t_2]}|M_{t_2}$ is a graph of $f|M_{t_2} : M_{t_2} \to \pi^{-1}(t_2)$.

Let π and f be written as the local forms (2.4) or (2.5). Then, it is known that the 1-parameter family of $f|M_t : M_t \to \pi^{-1}(t)$ is a birth or death bifurcation or an exchange of levels of the corresponding two critical values, respectively ($t \in [t_1,t_2]$; see [12]). Since it is easy to construct the required embeddings $F_{[t_1,t_2]} : M_{[t_1,t_2]} \to \pi^{-1}([t_1,t_2]) \times \mathbf{R}_1^2$ for both cases, we leave these constructions to the reader. This completes the proof. $\square$

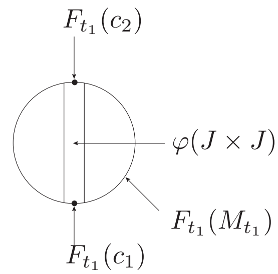

Figure 7. (a)

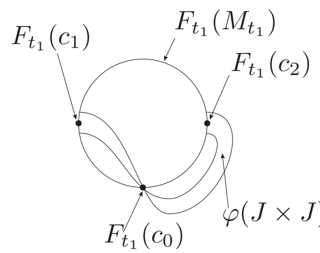

Figure 7. (b)

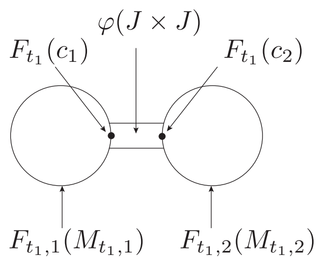

Figure 7. (c)

Remark 3.2. To prove Lemma 3.1, it is necessary that an embedding $F_{t_i}|M_{t_i} : M_{t_i} \to \pi^{-1}(t_i) \times \mathbf{R}^2_1$ is a graph of $f|M_{t_i} : M_{t_i} \to \pi^{-1}(t_i)$, $i = 1, 2$. The reason is as follows. Suppose that π and f are expressed as the local form (2.2) and the positive direction of $\mathbf{R}^1$ is right to left in Figure 1(b). We let $q \in S(f) \cap M_b$ be the singular point of f such that $f(q) = r$, $M_{[t_1,t_2],q}$ is the component of $M_{[t_1,t_2]}$ which contains q and $M_{t_1,q}$ is the boundary of $M_{[t_1,t_2],q}$. To construct an embedding lift $F_{[t_1,t_2]} : M_{[t_1,t_2]} \to \pi^{-1}([t_1, t_2]) \times \mathbf{R}^2_1$ of $f|M_{[t_1,t_2]} : M_{[t_1,t_2]} \to \pi^{-1}([t_1, t_2])$, it is necessary that $F_{t_1}(M_{t_1,q})$ and $F_{t_1}(M_{t_1} \setminus M_{t_1,q})$ are unlinked in $\pi^{-1}(t_1) \times \mathbf{R}^2_1$. Thus, the above assumption is necessary to prove Lemma 3.1.

Remark 3.3. After the author proved Theorem 1.1, Akhmetiev pointed out that any map $f : S^2 \to N$, where N is a closed orientable surface, is realizable in $\mathbf{R}^4$. For the definition of "realizable", see Section 1 and for the proof of this result, see [16, p. 148].

4. Examples

In this section, we will give two examples which clarify Theorem 1.1.

Example 4.1. Let $f_1 : S^2 \to \mathbf{R}^2$ be a stable map such that $f_1(S^2)$ is depicted as in Figure 8(a). See [11], [13] for the precise definition. Then Figure 8(b) shows how to construct the embedding lift $F_1 : S^2 \to \mathbf{R}^4$ of f_1 which is described in the proof of Theorem 1.1. In [11], Haefliger showed that for the above stable map f_1, there is no immersion $g_1 : S^2 \to \mathbf{R}^3$ such that $\pi_2^1 \circ g_1 = f_1$ is satisfied (see also [17]).

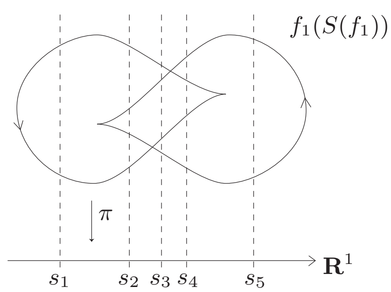

Figure 8. (a)

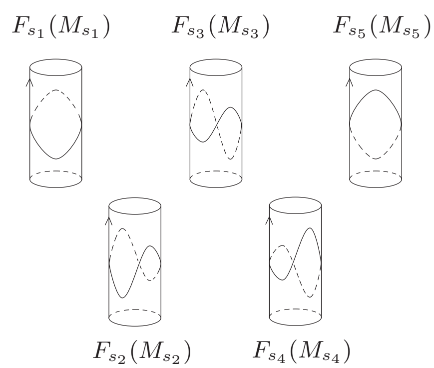

Figure 8. (b)

Example 4.2. Let $f_2 : \mathbf{R}P^2 \to \mathbf{R}^2$ be a stable map such that $f_2(\mathbf{R}P^2)$ is depicted as in Figure 9(a). See [13] for the precise definition. Then Figure 9(b) shows how to construct the embedding lift $F_2 : \mathbf{R}P^2 \to \mathbf{R}^4$ of f_2 which is described in the proof of Theorem 1.1. For the above stable map f_2, there exists an immersion $g_2 : \mathbf{R}P^2 \to \mathbf{R}^3$ such that $\pi_2^1 \circ g_2 = f_2$ (see Figure 9(c)). This immersion g_2 is known as the *Boy surface* and that there is no embedding lift $G_2 : \mathbf{R}P^2 \to \mathbf{R}^4$ such that $\pi_3^1 \circ G_2 = g_2$ is satisfied (see [7], [9], [19]).

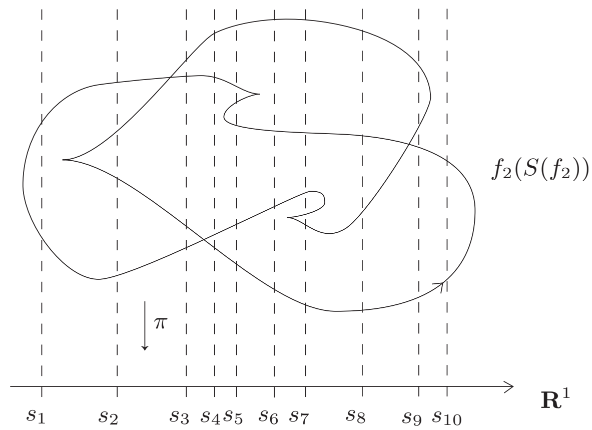

FIGURE 9. (a)

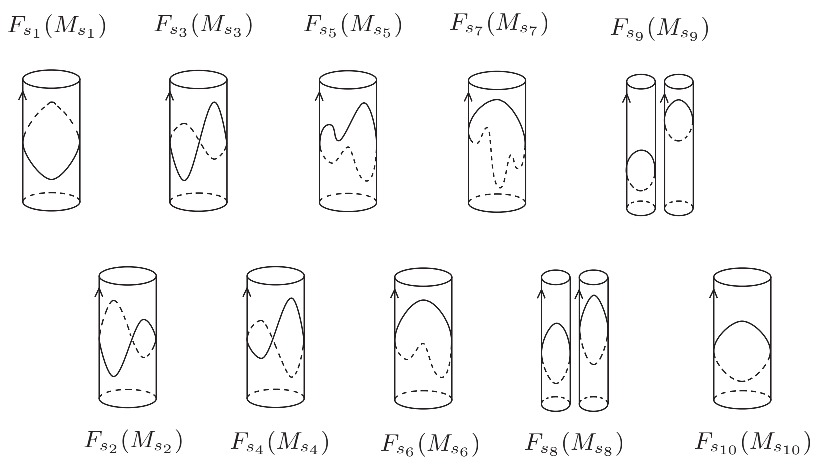

FIGURE 9. (b)

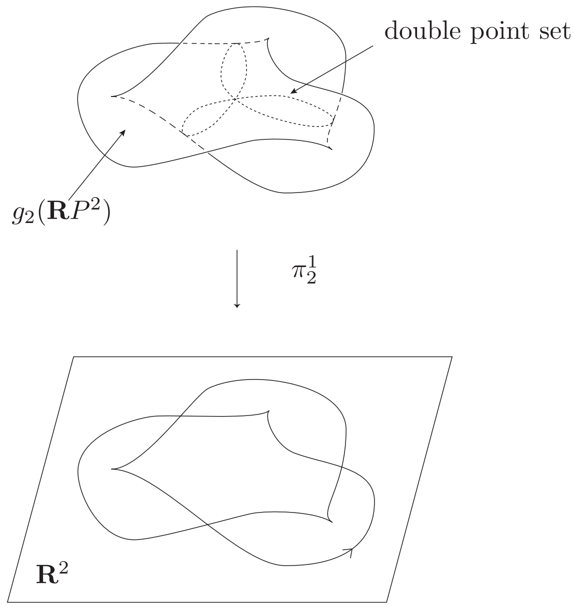

FIGURE 9. (c)

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